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1976 J. Phys. A: Math. Gen. 9 555

(http://iopscience.iop.org/0305-4470/9/4/012)

J. Phys. A: Math. Gen., Vol. 9, No. 4, 1976. Printed in Great Britain. © 1976

Cartan frames and rotating universes†

I Damião Soares
Centro Brasileiro de Pesquisas Físicas, Rio de Janeiro, Brasil

Received 9 July 1975, in final form 9 December 1975

Abstract. Starting from a given cosmological model, we rotate (in the sense described in the paper) Cartan moving frames over the manifold of the model to construct rotating cosmological models. A class of solutions with perfect fluid which correspond to rotating models is obtained, from Minkowski space.

Rotating universes have the interesting property that matter rotates with non-zero angular velocity, in a local inertial system in whose origin it is taken to be at rest at the moment considered (Gödel 1949). Such a rotation can be incorporated naturally in Cartan moving frames (Cartan 1922, see also Cartan 1952) on the manifold. This provides a simple geometrical tool for stopping or starting the rotation of a given universe and analysing the resulting model. As an example, we obtain by this process a class of rotating models, starting from Minkowski space.

The line element of any locally Lorentzian manifold can always be decomposed:

$$ds^2=(\theta^0)^2-(\theta^1)^2-(\theta^2)^2-(\theta^3)^2. \qquad (1)$$

To a decomposition of the ds^2 in squares of the form (1) there corresponds *in a unique way* six 1-forms $\omega_{AB}=-\omega_{BA}$, linear in θ^A and satisfying the structure equations‡

$$d\theta^A=-\omega^A{}_B\wedge\theta^B. \qquad (2)$$

The decomposition (1) defines in each point of the manifold a Cartesian (moving) frame of reference, with θ^A being the components of the instantaneous translation and ω_{AB} the components of the instantaneous rotation of this frame.§ An observer having Cartesian coordinates (X^A) with respect to the moving frame is at rest for such an infinitesimal motion of the moving frame if

$$dX^A+\theta^A+\omega^A{}_BX^B=0. \qquad (3)$$

Now let us consider a stationary rotating universe and the local inertial frame of an observer co-moving with matter. A fluid particle with coordinates X^A can be at rest with respect to the frame if its rotation is assimilated—in the sense of (3)—to an additional instantaneous rotation of the frame. We have then a naturally defined Cartan moving frame, where the additional ω

$$dX^A-\tilde{\omega}^A{}_BX^B=0$$

† Partially supported by CNPq/Brasil.
‡ Capital Latin indices run from 0 to 3; they are raised and lowered with the Minkowski metric η_{AB}, $\eta^{AB}=\mathrm{diag}(+1,-1,-1,-1)$.
§ An accessible reference to E Cartan's idea of moving frames may be found in chapter 3, H Cartan (1971).

are the 1-forms of the rotation of the universe. Once these are prescribed or identified on a given model, we can introduce or eliminate rotation by adding or subtracting terms in (3) of the above type, and modifying structure equations correspondingly.

The rotation 1-forms can be obtained as follows. An observer co-moving with matter has four velocity.

$$u^A = \delta_0^A, \qquad u_A = \delta_A^0 \tag{4}$$

in a local moving frame determined by (1). This corresponds to a matter–velocity field

$$u^\mu = e^\mu_{(A)} u^A = e^\mu_{(0)}$$

where the tetrads $e^\mu_{(A)}$ are defined by

$$\theta^A = e^{(A)}_\mu \, dX^\mu. \tag{5}$$

In the local frame, the rotation of the world lines of matter

$$\Omega_{AB} = (u_{\mu\|\nu} - u_{\nu\|\mu}) e^\mu_{(A)} e^\nu_{(B)}$$

is given by

$$\Omega_{AB} = \gamma_{OAB} - \gamma_{OBA} \tag{6}$$

in which we used the Ricci rotation coefficients defined by

$$\gamma_{ABC} = e_{(A)\beta\|\gamma} e^\beta_{(B)} e^\gamma_{(C)}. \tag{7}$$

Equation (6) can be expressed as the 2-form

$$\Omega = \Omega_{AB} \theta^A \wedge \theta^B \tag{8}$$

of the rotation of the universe, and by (2) we have the relation

$$\tfrac{1}{2}\Omega = d\theta^0. \tag{9}$$

In general, the rotation of a cosmological model is zero if and only if $\Omega = 0$. More properly, a rotation in the (X^i, X^j) plane, $i, j = 1, 2, 3$, of a local inertial observer will only contribute to Ω in components of $\theta^i \wedge \theta^j$.†

It is then clear that the most general type of rotation we can introduce in a manifold is given by the new 1-forms

$$\bar{\omega}_{AB} = \omega_{AB} + \sum_c \alpha_{ABC}(X) \epsilon_{ABC} \theta^c \tag{10}$$

$$A, B, C = (0, i, j), \qquad i, j = 1, 2, 3.$$

ϵ_{ABC} is the Levi–Civitta symbol and ω_{AB} are the rotation 1-forms of the original manifold. Equation (10) defines uniquely the structure of a manifold which is a rotating cosmological model. The new θ will be solutions of $d\theta^A = -\bar{\omega}^A{}_B \wedge \theta^B$. Besides, (10) must satisfy Einstein's equation for a given source, and this results in equations relating the α and the matter content of the model. The method is general and the geometrical interpretation obvious.

† For instance, we can verify that $d\theta^0 = \alpha(t)\theta^0 \wedge \theta^1$ does not correspond to a rotation but to a local acceleration along X^1. Such universes contain a class of solutions of Einstein's equation (for $d\theta^1 = d\theta^2 = d\theta^3 = 0$) known as Ehlers–Kundt plane-gravitational waves.

We will now consider an example. We start with a Minkowski space and, according to (10), proceed to rotate the (X^1, X^2) plane of the Cartan frame as

$$\omega_{01}=\alpha\theta^2, \qquad \omega_{02}=\beta\theta^1, \qquad \omega_{12}=\gamma\theta^0 \tag{11}$$

with α, β, γ as constants. Using (11) in (2) we obtain

$$\begin{aligned} d\theta^0 &= (\alpha-\beta)\theta^1\wedge\theta^2 \\ d\theta^1 &= (\alpha+\gamma)\theta^2\wedge\theta^0 \\ d\theta^2 &= -(\beta-\gamma)\theta^1\wedge\theta^0. \end{aligned} \tag{12}$$

We assume an observer co-moving with matter as in (4). The congruence of world lines of matter is defined by the unity velocity field $u^\mu = e^\mu_{(0)}$. It is geodesic and expansion free if

$$\gamma^0{}_{A0}=0 \tag{13}$$

and

$$\gamma^{0A}{}_A=0 \tag{14}$$

respectively. Conditions (13) and (14) are always satisfied for choice (10), provided the original ω satisfy them.

For a perfect fluid source, the density of matter and the pressure, as measured by the above co-moving observers, are denoted by ρ and P respectively. The vanishing of the divergence of the energy–momentum tensor, (13) and (14) imply that ρ and P are constants, as expected. Einstein's equations with a cosmological term

$$R_{AB}-(\tfrac{1}{2}R-\Lambda)\eta_{AB}=K[(\rho+P)u_Au_B-P\eta_{AB}] \tag{15}$$

are satisfied by (12) for

$$\gamma=0 \qquad \begin{aligned} 2\alpha\beta+\Lambda &= K\rho \\ 2\Lambda &= K(\rho-P) \end{aligned} \tag{16}$$

with arbitrary equation of state. Notice the two special cases:

(i) Incoherent matter

$$\begin{aligned} \gamma&=0 \qquad 4\alpha\beta=K\rho=2\Lambda. \\ P&=0 \end{aligned} \tag{17}$$

(ii) Extreme relativisitic perfect fluid

$$\begin{aligned} \gamma&=0 \qquad 2\alpha\beta=K\rho \\ \Lambda&=0 \qquad P=\rho. \end{aligned} \tag{18}$$

All solutions correspond to homogeneous rotating cosmological models. The line element will be given by (1) with the θ satisfying (12)—integrability conditions for this case

$$C^F{}_{[AC}C^B_{D]F}=0$$

are automatically satisfied and guarantee the existence of the θ, where the structure constants $C^A{}_{BC}$ are defined by

$$d\theta^A=-\tfrac{1}{2}C^A{}_{BC}\theta^B\wedge\theta^C.$$

New $\tilde{\theta}$ may be defined by

$$\theta^0 = A_0\tilde{\theta}^0 \qquad \theta^1 = A_1\tilde{\theta}^1 \qquad \theta^2 = A_2\tilde{\theta}^2 \qquad \theta^3 = A_3\tilde{\theta}^3$$

and used to transform (12) into

$$\begin{aligned} &d\tilde{\theta}^0 = \epsilon_0\tilde{\theta}^1 \wedge \tilde{\theta}^2 \qquad d\tilde{\theta}^1 = \epsilon_1\tilde{\theta}^2 \wedge \tilde{\theta}^0 \\ &d\tilde{\theta}^2 = \epsilon_2\tilde{\theta}^1 \wedge \tilde{\theta}^0 \end{aligned} \tag{19}$$

where, for solutions (16)

$$\begin{aligned} &\epsilon_1 = -\alpha(A_0A_2/A_1) \\ &\epsilon_2 = -\beta(A_0A_1/A_2) \\ &\epsilon_0A_0^2 = \epsilon_2A_2^2 - \epsilon_1A_1^2. \end{aligned} \tag{20}$$

If we consider cases (17) and (18) only, we must have $\alpha\beta > 0$ because $\rho > 0$. This implies

$$\epsilon_1\epsilon_2 > 0$$

and we have the following groups on sections $X^3 = \text{constant}$:

ϵ_0	$\epsilon_1 = \epsilon_2$	$a = \lvert A_1/A_2 \rvert$
± 1	∓ 1	>1
± 1	± 1	<1
0	1	1

The two first cases are equivalent for the change of θ^1 and θ^2. They correspond to Bianchi type VIII (Ellis and MacCallum 1969).

In general for the case (16), condition $\alpha\beta > 0$ is not necessary and (12) or (19) allows for other group types on $X^3 = \text{constant}$ sections and correspondingly more solutions.

To illustrate, consider the case when

$$\epsilon_0 = -1, \qquad \epsilon_1 = \epsilon_2 = +1, \qquad a > 0. \tag{21}$$

The line element is given by

$$ds^2 = A_0^2(\tilde{\theta}^0)^2 - A_1{}^2(\tilde{\theta}^1)^2 - A_2{}^2(\tilde{\theta}^2)^2 - A_3{}^2(\tilde{\theta}^3)^2 \tag{22}$$

with

$$\begin{aligned} &\tilde{\theta}^0 = -\sin x^1\, dx^2 + e^{x^2}\cos x^1\, dt \qquad &\tilde{\theta}^1 = dx^1 + e^{x^2}\, dt \\ &\tilde{\theta}^2 = \cos x^1\, dx^2 + e^{x^2}\sin x^1\, dt \qquad &\tilde{\theta}^3 = dx^3 \end{aligned}$$

which satisfy (19) for (21), and

$$\begin{aligned} &K\rho = 4/A_0^2 = 2\Lambda \qquad \text{for } P = 0 \\ &K\rho = 2/A_0^2 \qquad \text{for } P = \rho \\ &A_0{}^2 = A_1{}^2 - A_2{}^2. \end{aligned} \tag{23a}$$

Matter propagates along the geodesic congruence defined by the unit geodesic vector field

$$u^{\mu} = (1/A_0)(e^{-x^2} \cos x^1, -\cos x^1, -\sin x^1, 0). \tag{23b}$$

Solution (23) does not include the Gödel model as a particular case.

For the value $a = 1$, we have the *limiting* case $\epsilon_0 = 0$ so that we have a finite density of matter. The group acting on X^3 = constant section is Bianchi I—this corresponds to the homogeneous non-rotating anisotropic solution with dust,

$$ds^2 = A_0{}^2\, dt^2 - A_1{}^2\, e^{-2t}(dx^1)^2 - A_1{}^2\, e^{2t}(dx^2)^2 - (dx^3)^2$$

$$K\rho = 4/A_0^2 = 2\Lambda.$$

The method could also be used to eliminate the rotation of a rotating universe. We have done this for the Gödel model, obtaining a non-rotating model which is a solution of Einstein's equations for a fluid with density ρ and anisotropic pressure. The cosmological constant has a sign opposite to that of Gödel's, and the components of pressure cannot all have the same sign simultaneously, even though satisfying energy conditions. A class of two-dimensional space-like sections have peculiar properties related to the completeness of geodesics, which we intend to discuss in the future. At present, we are working on a rotating expanding model which is a demanding application of the method.

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